Homework 6 Solutions

ECON 441: Introduction to Mathematical Economics — Instructor: Div Bhagia

Exercise 10.5

1. (a) $2e^{2t+4}$

   (b) $-9e^{1-9t}$

   (c) $2te^{t^2+1}$

   (d) $-10te^{2-t^2}$

   (e) $(2ax+b)e^{ax^2+bx+c}$

   (f) $\dfrac{dy}{dx} = x\dfrac{d}{dx}e^x + e^x\dfrac{dx}{dx} = xe^x + e^x = (x+1)e^x$

   (g) $\dfrac{dy}{dx} = x^2\left(2e^{2x}\right) + 2xe^{2x} = 2x(x+1)e^{2x}$

   (h) $\dfrac{dy}{dx} = a\left(xbe^{bx+c} + e^{bx+c}\right) = a(bx+1)e^{bx+c}$

3. (a) $\dfrac{dy}{dt} = \dfrac{35t^4}{7t^5} = \dfrac{5}{t}$

   (b) $\dfrac{dy}{dt} = \dfrac{act^{o-1}}{at^c} = \dfrac{c}{t}$

   (c) $\dfrac{dy}{dt} = \dfrac{1}{t+19}$

   (d) $\dfrac{dy}{dt} = 5\dfrac{2(t+1)}{(t+1)^2} = \dfrac{10}{t+1}$

   (e) $\dfrac{dy}{dx} = \dfrac{1}{x} - \dfrac{1}{1+x} = \dfrac{1}{x(1+x)}$

   (f) $\dfrac{dy}{dx} = \dfrac{d}{dx}[\ln x + 8\ln(1-x)] = \dfrac{1}{x} + \dfrac{-8}{1-x} = \dfrac{1-9x}{x(1-x)}$

   (g) $\dfrac{dy}{dx} = \dfrac{d}{dx}[\ln 2x - \ln(1+x)] = \dfrac{2}{2x} - \dfrac{1}{1+x} = \dfrac{1}{x(1+x)}$

(h) $\frac{dy}{dx} = 5x^4 \frac{2x}{x^2} + 20x^3 \ln x^2 = 10x^3 \left(1 + 2\ln x^2\right) = 10x^3(1 + 4\ln x)$

7. (a) Since $\ln y = \ln 3x - \ln(x+2) - \ln(x+4)$, we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} - \frac{1}{x+2} - \frac{1}{x+4} = \frac{8 - x^2}{x(x+2)(x+4)}$$

Hence,

$$\frac{dy}{dx} = \frac{8 - x^2}{x(x+2)(x+4)} \cdot \frac{3x}{(x+2)(x+4)} = \frac{3\left(8 - x^2\right)}{(x+2)^2(x+4)^2}$$

(b) Since $\ln y = \ln\left(x^2 + 3\right) + x^2 + 1$, we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2 + 3} + 2x = \frac{2x\left(x^2 + 4\right)}{x^2 + 3}$$

Hence,

$$\frac{dy}{dx} = \frac{2x\left(x^2 + 4\right)}{x^2 + 3}\left(x^2 + 3\right) e^{x^2+1} = 2x\left(x^2 + 4\right) e^{x^2+1}$$

Exercise 7.4

1. (a) $y = 2x_1^3 - 11x_1^2 x_2 + 3x_2^2$

$$\frac{dy}{dx_1} = 6x_1^2 - 22x_1 x_2, \qquad \frac{dy}{dx_2} = -11x_1^2 + 6x_2$$

(d) $y = \frac{5x_1 + 3}{x_2 - 2}$

$$\frac{dy}{dx_1} = \frac{5(x_2 - 2) - (5x_1 + 3)\cdot 0}{(x_2 - 2)^2} = \frac{5}{x_2 - 2}$$

$$\frac{dy}{dx_2} = \frac{0 \cdot (x_2 - 2) - 1 \cdot (5x_1 + 3)}{(x_2 - 2)^2} = \frac{-(5x_1 + 3)}{(x_2 - 2)^2}$$

2.& 3. (a) $f(x,y) = x^2 + 5xy - y^3$

$$f_x = 2x + 5y \quad \rightarrow f_x(1,2) = 12$$
$$f_y = 5x - 3y^2 \quad \rightarrow f_y(1,2) = -7$$

(b) $f(x,y) = \left(x^2 - 3y\right)(x-2)$

$$f_x = (2x)(x-2) + \left(x^2 - 3y\right) \cdot 1$$
$$= 2x^2 - 4x + x^2 - 3y = 3x^2 - 4x - 3y$$

Then $f_x(1,2) = 3 - 4 - 6 = -7$

$$f_y = -3(x-2) + \left(x^2 - 3y\right) \cdot 0 = -3x + 6$$

Then $f_y(1,2) = -3 + 6 = 3.$

5. $U = U(x_1, x_2) = (x_1 + 2)^2 (x_2 + 3)^3$

(a)

$$U_1(x_1, x_2) = 2(x_1 + 2)(x_2 + 3)^3 = \frac{2U}{x_1 + 2}$$
$$U_2(x_1, x_2) = 3(x_1 + 2)^2 (x_2 + 3)^2 = \frac{3U}{x_2 + 3}$$

(b) $U_1(3,3) = 2(3+2)(3+3)^3 = 2 \times 5 \times 6^3 = 2160$

7. (a)

$$f(x,y,z) = x^2 + y^3 + z^4$$
$$f_x = 2x$$
$$f_y = 3y^2$$
$$f_z = 4z^3$$
$$\nabla f(x,y,z) = \begin{bmatrix} 2x \\ 3y^2 \\ 4z^3 \end{bmatrix}$$

(b)

$$
\begin{aligned}
f(x,y,z) &= xyz \\
f_x &= yz \\
f_y &= xz \\
f_z &= xy \\
\nabla f(x,y,z) &= \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix}
\end{aligned}
$$

Exercise 8.1

1. (a) $y = -x^3 - 3x$

$$dy = \left(-3x^2 - 3\right) dx = -3\left(x^2 + 1\right) dx$$

4. $Q = kp^{-n}, k > 0, n > 0$

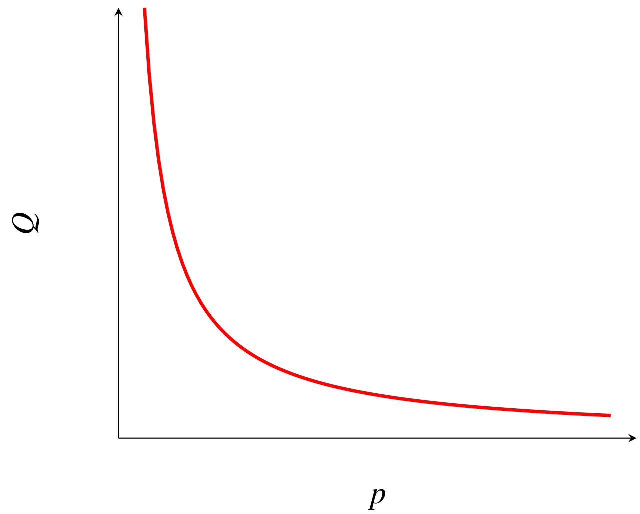
(a)

$$\frac{dQ}{dp} = -nkp^{-n-1}$$

$$\varepsilon_d = \frac{dQ}{dp} \cdot \frac{p}{Q} = \frac{-nkp^{-n-1} \cdot p}{kp^{-n}} = -n$$

No, the elasticity does not depend on the price.

(b) When $n = 1, \quad Q = \frac{k}{p}, \varepsilon_d = -1$



6. $Q = 100 - 2P + 0.02Y$

$$P = 20, Y = 5000 \longrightarrow Q = 100 - 40 + 100 = 160$$

(a)

$$\frac{dQ}{dP} \cdot \frac{P}{Q} = -2 \cdot \frac{20}{160} = -0.25$$

(b)

$$\frac{dQ}{dY} \cdot \frac{Y}{Q} = 0.02 \cdot \frac{5000}{160} = 0.625$$

Exercise 8.2

3. (a)

$$y = \frac{x_1}{x_1 + x_2}$$

$$\frac{dy}{dx_1} = \frac{1\,(x_1 + x_2) - 1 \cdot x_1}{(x_1 + x_2)^2} = \frac{x_2}{(x_1 + x_2)^2}$$

$$\frac{dy}{dx_2} = \frac{0 \cdot (x_1 + x_2) - 1x_1}{(x_1 + x_2)^2} = \frac{-x_1}{(x_1 + x_2)^2}$$

$$dy = \frac{x_2}{(x_1 + x_2)^2} \cdot dx_1 - \frac{x_1}{(x_1 + x_2)^2} dx_2$$

4. $Q = a + bP^2 + R^{1/2} \quad (a < 0, b > 0)$

$$\varepsilon_{QP} = \frac{dQ}{dP} \cdot \frac{P}{Q} = \frac{2bP \cdot P}{Q} = \frac{2bP^2}{a + bP^2 + R^{1/2}}$$

$$\varepsilon_{QR} = \frac{dQ}{dR} \cdot \frac{R}{Q} = \frac{1}{2} R^{\frac{1}{2}-1} \cdot \frac{R}{Q} = \frac{R^{1/2}}{2\left(a + bP^2 + R^{1/2}\right)}$$

5.

$$\frac{d\varepsilon_{QP}}{dP} = \frac{4bP\left(a + bP^2 + R^{1/2}\right) - 2bP\left(2bP^2\right)}{\left(a + bP^2 + R^{1/2}\right)^2}$$

$$= \frac{4bP\left(a + R^{1/2}\right)}{\left(a + bP^2 + R^{1/2}\right)^2}$$

Denominator is positive. Numerator is positive when $a + R^{1/2} > 0$. So $\frac{d\varepsilon_{QP}}{dP} \geq 0$ when $a + R^{1/2} \geq 0$ and $\frac{d\varepsilon_{QP}}{dP} < 0$ when $a + R^{1/2} < 0$.

$$\frac{d\varepsilon_{QR}}{dR} = \frac{-bP^2 R^{-1/2}}{\left(a + bP^2 + R^{1/2}\right)^2} < 0$$

$$\frac{d\varepsilon_{QR}}{dP} = \frac{-bPR^{-1/2}}{\left(a + bP^2 + R^{1/2}\right)^2} < 0$$

$$\frac{d\varepsilon_{Q,R}}{dR} = \frac{R^{-1/2}\left(a + bP^2 + R^{1/2}\right) - R^{-1/2}R^{1/2}}{4\left(a + bP^2 + R^{1/2}\right)^2}$$
$$= \frac{R^{-1/2}\left(a + bP^2\right)}{4\left(a + bP^2 + R^{1/2}\right)^2}$$

Similar reasoning as before, $\frac{d\varepsilon_{QR}}{dR} \geq 0$ if $a + bp^2 \geq 0$ and $\frac{d\varepsilon_{QR}}{dR} < 0$ if $a + bp^2 < 0$.

6. $x = y_f^{1/2} + p^{-2}$

$$\varepsilon_{xp} = \frac{dx}{dp} \cdot \frac{p}{x} = \frac{-2p^{-2}}{y_f^{1/2} + p^{-2}} = \frac{-2}{y_f^{1/2}p^2 + 1} < 0$$

7. (a) $U = 7x^2y^3$

$$du = u_x dx + u_y \cdot dy$$
$$= 14xy^3 \cdot dx + 21x^2y^2 \cdot dy$$
$$= 7xy^2(2y \cdot dx + 3x \cdot dy)$$

(f) $U = (x - 3y)^3$

$$du = 3(x - 3y)^2 dx - 3(x - 3y)^2(-3)$$
$$= 3(x - 3y)^2(dx - 3dy)$$

Exercise 8.4

2. (a)

$$\begin{aligned} z &= f(x,y) = x^2 - 8xy - y^3 \\ x &= 3t \\ y &= 1 - t \end{aligned}$$

$$\begin{aligned} \frac{dz}{dt} &= f_x \frac{dx}{dt} + f_y \cdot \frac{dy}{dt} \\ &= (2x - 8y)3 + \left(-8x - 3y^2\right)(-1) \\ &= 6x - 24y + 8x + 3y^2 \\ &= 42t - 24(1-t) + 3(1-t)^2 \\ &= 3t^2 + 60t - 21 \end{aligned}$$

(b)

$$\begin{aligned} z &= f(u,v,t) = 7u + vt \\ u &= 2t^2, v = t + 1 \end{aligned}$$

$$\begin{aligned} \frac{dz}{dt} &= f_u \frac{du}{dt} + f_v \cdot \frac{dv}{dt} + f_t \frac{dt}{dt} \\ &= 7(4t) + t.1 + v \\ &= 28t + t + t + 1 \\ &= 30t + 1 \end{aligned}$$

(c)

$$\begin{aligned} z &= f(x,y,t) \\ x &= a + bt \\ y &= c + Rt \end{aligned}$$

$$\begin{aligned} \frac{dz}{dt} &= f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_t \\ &= bf_x + Rf_y + f_t \end{aligned}$$

4. (a)

$$
\begin{aligned}
w &= f(x, y, u) = ax^2 + bxy + cu \\
x &= \alpha u + \beta v \\
y &= \gamma u
\end{aligned}
$$

$$
\begin{aligned}
\frac{dw}{du} &= fx\frac{dx}{du} + fy \cdot \frac{dy}{du} + f_u \\
&= (2ax + by)\alpha + \gamma bx + c \\
&= (2a + \gamma b)x + \alpha by + c \\
&= (2a + \gamma b)(\alpha u + \beta v) + \gamma\alpha bu + c
\end{aligned}
$$

(b)

$$
\begin{aligned}
w &= f\left(x_1, x_2\right) \\
x_1 &= 5u^2 + 3v \\
x_2 &= u - 4v^3
\end{aligned}
$$

$$
\begin{aligned}
\frac{dw}{du} &= f_1\frac{dx_1}{du} + f_2\frac{dx_2}{du} \\
&= f_1 \cdot 10u + f_2 = 10uf_1 + f_2
\end{aligned}
$$

$$
\begin{aligned}
\frac{dw}{dv} &= f_1\frac{dx_1}{dv} + f_2\frac{dx_2}{dv} \\
&= f_1 \cdot 3 + f_2\left(-12v^2\right) = 3f_1 - 12v^2 f_2
\end{aligned}
$$