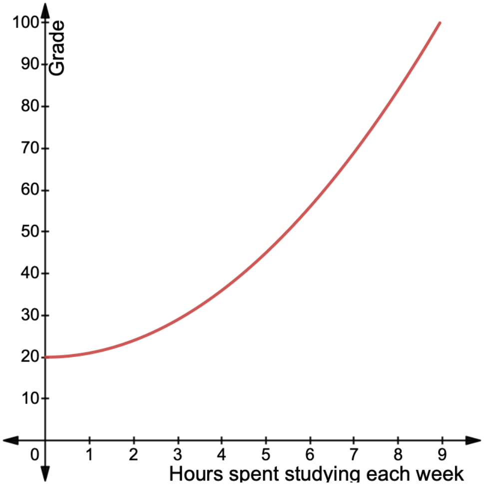
ECON 441

Introduction to Mathematical Economics

Div Bhagia

Lecture 5: Calculus

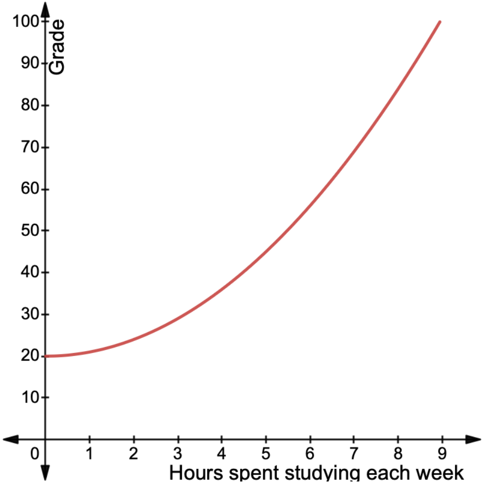
(Hypothetical) Production Function for Grades



$$y = f(x) = x^2 + 20$$

How much can your grade can increase if you study for one additional hour per week?

(Hypothetical) Production Function for Grades



$$y = f(x) = x^2 + 20$$

How much can your grade can increase if you study for one additional hour per week? Depends on how much you are studying right now!

Average Rate of Change

Use Δ to denote change:

$$\Delta x = x_1 - x_0$$

Change in y per unit change in x:

$$\frac{\Delta y}{\Delta x} = \frac{f\left(x_0 + \Delta x\right) - f\left(x_0\right)}{\Delta x}$$

Average Rate of Change

$$y = f(x) = x^2 + 20$$

What happens if you go from 6 to 8 hours of studying?

$$x_0 = 6, x_1 = 8 \rightarrow \Delta x = x_1 - x_0 = 2$$

Total change in grades:

$$f(x_0 + \Delta x) - f(x_0) =$$

Per hour change in grade:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} =$$

The Derivative

Usually interested in minuscule changes from x_0.

The derivative of a function is defined as:

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Note that

$$\frac{\Delta y}{\Delta x} = \frac{f\left(x_0 + \Delta x\right) - f\left(x_0\right)}{\Delta x}$$

The Derivative

For the function: $y = f(x) = x^2 + 20$

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \\ &= \frac{(x_0 + \Delta x)^2 + 20 - (x_0^2 + 20)}{\Delta x} \\ &= \frac{x_0^2 + (\Delta x)^2 + 2x_0\Delta x - x_0^2}{\Delta x} \\ &= 2x_0 + \Delta x\end{aligned}$$

Then the derivative is given by:

$$\frac{dy}{dx} = f'(x_0) = \lim_{\Delta x \to 0} 2x_0 + \Delta x = 2x_0$$

The Derivative

The derivative:

$$\frac{dy}{dx} = f'(x_0) = \lim_{\Delta x \to 0} \frac{f\left(x_0 + \Delta x\right) - f\left(x_0\right)}{\Delta x}$$

Alternatively,

$$\frac{dy}{dx} = f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

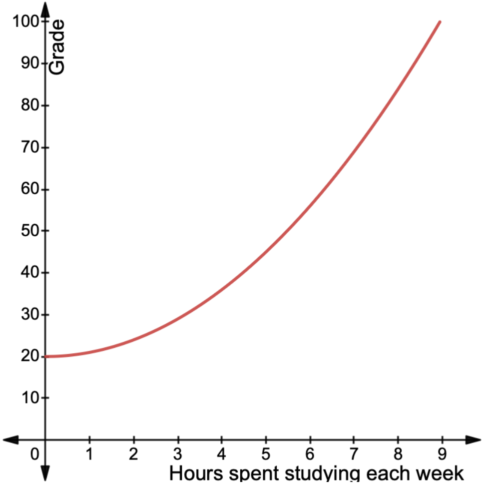
The Derivative

For the function: $y = f(x) = x^2 + 20$

$$\begin{aligned}\frac{dy}{dx} &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{x^2 + 20 - x_0^2 - 20}{x - x_0} \\ &= \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} \\ &= \lim_{x \to x_0} x + x_0 = 2x_0\end{aligned}$$

Derivative = Slope of the Tangent Line

$$y = f(x) = x^2 + 20 \rightarrow f'(x) = 2x$$



Concept of a limit

We say that L is the **limit** of $f(x)$ at a, i.e.

$$\lim_{x \to a} f(x) = L$$

if $f(x)$ approaches L as x approaches a from any direction.

Note that we don't actually set $x = a$.

Also for the limit to exist at a point we need the function to approach the same value from both directions.

Concept of a limit

Left-side limit: If x approaches a from the left side:

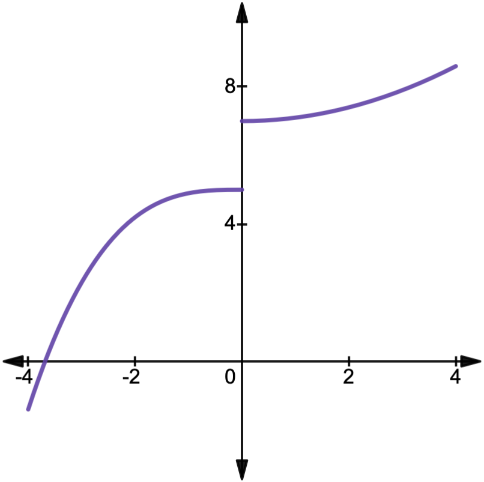
$$\lim_{x \to a^-} f(x)$$

Right-side limit: If x approaches a from the right side:

$$\lim_{x \to a^+} f(x)$$

Only when both left-side and right-side limits have a common finite value, we say that the limit exists.

Example: Limit doesn't exist



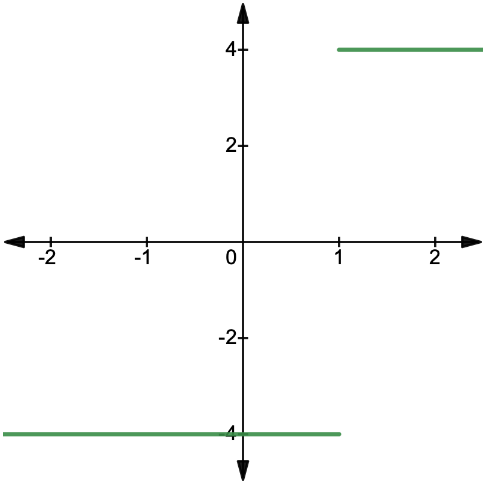
Limit: Another example

$$f(x) = \frac{4x-4}{|x-1|}$$

$$\lim_{x \to 1^-} f(x) =$$

$$\lim_{x \to 1^+} f(x) =$$

Limit: Another example

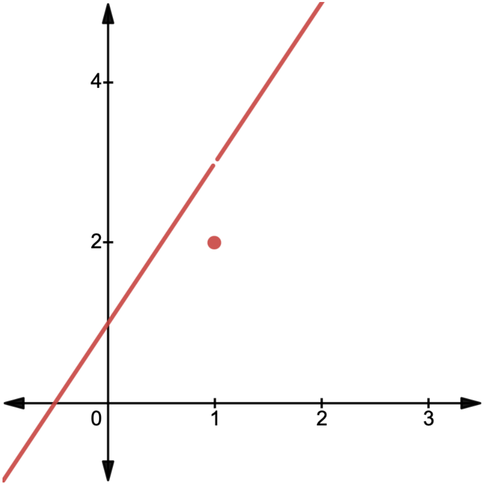


Continuity of a Function

A function $y = f(x)$ is said to be continuous at a if $\lim_{x \to a} f(x)$ exists and

$$\lim_{x \to a} f(x) = f(a)$$

Discontinuity: Example



$$y = \begin{cases} 2x + 1 \text{ if } x < 1 \\ 2 \text{ if } x = 1 \\ 2x + 1 \text{ if } x > 1 \end{cases}$$

In this example, the limit exists but the function is not continuous.

Differentiability and Continuity

$f'(x_0)$ exists if the following limit exists:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

A function $y = f(x)$ is continuous at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Any connection?

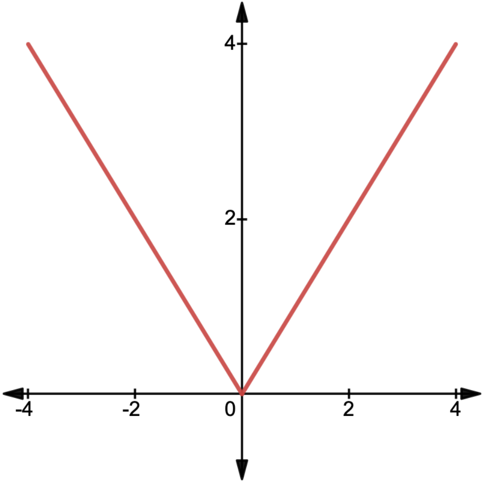
Differentiability and Continuity

Continuity is a necessary condition for differentiability, but it is not sufficient.

f is not continuous $\implies$ f is not differentiable

f is continuous $\implies$ f could be differentiable or not

Continuous but not differentiable



$$y = |x|$$

$$y = \begin{cases} x \text{ if } x \geqslant 0 \\ -x \text{ if } x < 0 \end{cases}$$

This function is continuous but not differentiable.

So how to differentiate functions?

Rules of differentiation, easier than taking the limit each time

Constant function rule:
For function $f(x) = k$, $f'(x) = 0$.

Power function rule:
For function $f(x) = x^n$, $f'(x) = nx^{n-1}$.

Generalized power function rule:
For function $f(x) = cx^n$, $f'(x) = cnx^{n-1}$.

Rules of Differentiation

Two or more functions of one variable

Sum-Difference Rule

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

Rules of Differentiation

Two or more functions of one variable

Quotient Rule

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Inverse Function Rule

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

Rules of Differentiation

Functions of Different Variables

Chain Rule

For $z = f(y), \quad y = g(x)$

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = f'(y)g'(x)$$

References and Homework

Textbook Reference: Sections 6.2-6.4, 6.7, 7.1-7.3

Homework Questions:

- Exercise 6.2: 2,3
- Exercise 7.1: 3
- Exercise 7.2: 3 (d) (e), 7, 8
- Exercise 7.3: 1-6