Rules of Differentiation

ECON 441: Introduction to Mathematical Economics — Instructor: Div Bhagia

Function of a Single Variable

- <u>Constant function rule:</u> If $f(x) = k$, $f'(x) = 0$.
- <u>Power function rule:</u> If $f(x) = x^n$, $f'(x) = nx^{n-1}$.
- <u>Generalized power function rule:</u> If $f(x) = cx^n$, $f'(x) = cnx^{n-1}$.

  Example. For $y = f(x) = 3x^2$, $f'(x) = 6x$.
- <u>Inverse function rule:</u> Given $y = f(x)$ and inverse function $x = f^{-1}(y)$

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

Exercises

1. Find the derivative of $y = -3x^6$?

2. Verify the inverse function rule for $y = 6x + 1$.

3. Find the derivative of $y = 1/x$.

In the class, we learned the limit definition of the derivative. In particular,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

All the rules of differentiation can actually be derived from the limit definition of the derivative. But it is easier to have a set of rules to quickly differentiate a function rather than writing down the limit each time.

For example, if we have the function $y = ax$, using the limit definition of the derivative:

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{a(x + \Delta x) - ax}{\Delta x} \\ &= \lim_{\Delta x \to 0} a = a\end{aligned}$$

Similarly, for $y = ax^2$

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{a(x + \Delta x)^2 - ax^2}{\Delta x} \\ &= \frac{ax^2 + a\Delta x^2 + 2ax\Delta x - ax^2}{\Delta x} \\ &= a\Delta x + 2ax\end{aligned}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2ax$$

Exercise. Verify the constant function rule using the limit definition.

Two or More Functions of a Single Variable

- Sum-Difference Rule

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

Example. For $y = 2x + x^2 - x^3$, the derivative is given by $\frac{dy}{dx} = 2 + 2x - 3x^2$

- Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

Example. For $y = 3x\left(x^2 + 1\right)$

$$\begin{aligned}\frac{dy}{dx} &= 3x(2x) + 3\left(x^2 + 1\right)\\ &= 6x^2 + 3x^2 + 3 = 9x^2 + 3\end{aligned}$$

Alternatively, one can write $y = 3x^3 + 3x$ and evaluate the derivative directly.

- Quotient Rule

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Example. Given the function

$$y = \frac{2x - 3}{x + 1}$$

We can calculate the derivative using the quotient rule as follows:

$$\begin{aligned}\frac{dy}{dx} &= \frac{2(x + 1) - (2x - 3)1}{(x + 1)^2}\\ &= \frac{2x + 2 - 2x + 3}{(x + 1)^2} = \frac{5}{(x + 1)^2}\end{aligned}$$

Exercise. Calculate the derivative for the following functions using product or quotient rule:

$$y = x(x+1), \quad y = \frac{1}{x}, \quad y = \frac{2x^3 - x}{x^2}$$

Functions of Different Variables

Chain Rule

If we have two functions:

$$z = f(y), \quad y = g(x)$$

Then,

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = f'(y)g'(x)$$

Example. $R = pQ$ is revenue from the sale of quantity Q at price p. Quantity produced $Q = aL$ depends on labor input L. Then,

$$\frac{dR}{dL} = \frac{dR}{dQ} \cdot \frac{dQ}{dL} = p \cdot a$$

Exercise. Find the derivative of f with respect to x where:

$$f(y) = y^2 - 1 \qquad y = 2x^2$$

Exercise. Find the derivative of $y = \dfrac{1}{(2x+1)^3}$ using Chain Rule by defining the outer function $y = f(u) = \dfrac{1}{u^3}$ and inner function $u = g(x) = 2x + 1$.

If you are bored:

Exercise. Verify the product rule using the limit definition of the derivative.

Easier if you start with the following definition:

$$h'(x) = \lim_{x_0 \to x} \frac{h(x) - h(x_0)}{x - x_0}$$

(Hint: Start by adding and subtracting $f(x)g(x_0)$ in the numerator.)