Homework 3 Solutions

ECON 441: Introduction to Mathematical Economics — Instructor: Div Bhagia

Exercise 5.1

3. (a) Yes (b) Yes
   (c) Yes (d) No, the second row = 2 × first row.

4. Yes, we get the same answer.

5. (a)

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 9 \\ -1 & 0 & 0 \end{bmatrix}$$

Exchange Row 2 and 3

$$A_1 = \begin{bmatrix} 1 & 5 & 1 \\ -1 & 0 & 0 \\ 0 & 3 & 9 \end{bmatrix}$$

New Row 2 = Row 1 + Row 2

$$A_2 = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 5 & 1 \\ 0 & 3 & 9 \end{bmatrix}$$

New Row 3 = Row 3–3/5 × Row 2

$$A_3 = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & \frac{42}{5} \end{bmatrix}$$

The resulting echelon matrix A_3 contains 3 nonzero rows and hence A has rank 3. Since A is full-rank, it is a nonsingular matrix.

(b)

$$B = \begin{bmatrix} 0 & -1 & -4 \\ 3 & 1 & 2 \\ 6 & 1 & 0 \end{bmatrix}$$

New Row 1 = Row 2, New Row 2 = Row 3, New Row 3 = Row 1

$$B_1 = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 1 & 0 \\ 0 & -1 & -4 \end{bmatrix}$$

New Row 2 = Row 2–2 $\times$ Row 1

$$B_2 = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & -4 \\ 0 & -1 & -4 \end{bmatrix}$$

New Row 3 = Row 3–Row 2

$$B_3 = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

The resulting echelon matrix B_3 contains only 2 nonzero rows and hence B has rank 2. Since the rank of B is less than the number of rows and columns, B is a singular matrix.

(c)

$$C = \begin{bmatrix} 7 & 6 & 3 & 3 \\ 0 & 1 & 2 & 1 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

Interchange rows 2 and 3.

$$C_1 = \begin{bmatrix} 7 & 6 & 3 & 3 \\ 8 & 0 & 0 & 8 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

New Row 2 = Row 2$-\frac{8}{7}\times$Row 1

$$C_2 = \begin{bmatrix} 7 & 6 & 3 & 3 \\ 0 & -\frac{48}{7} & -\frac{24}{7} & 8-\frac{24}{7} \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 7 & 6 & 3 & 3 \\ 0 & -\frac{48}{7} & -\frac{24}{7} & \frac{32}{7} \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

New Row 3 = Row 3$+\frac{7}{48}$ Row 2

$$C_3 = \begin{bmatrix} 7 & 6 & 3 & 3 \\ 0 & -\frac{48}{7} & -\frac{24}{7} & \frac{32}{7} \\ 0 & 0 & \frac{3}{2} & \frac{5}{3} \end{bmatrix}$$

Rank of C is 3. The concept of nonsingularity is only defined for square matrices.

(d)

$$D = \begin{bmatrix} 2 & 7 & 9 & -1 \\ 1 & 1 & 0 & 1 \\ 0 & 5 & 9 & -3 \end{bmatrix}$$

New Row 2= Row 2$-0.5\times$ Row 1

$$D_1 = \begin{bmatrix} 2 & 7 & 9 & -1 \\ 0 & -2.5 & -4.5 & 1.5 \\ 0 & 5 & 9 & -3 \end{bmatrix}$$

New Row 3= Row 3 +2× Row 2

$$D_2 = \begin{bmatrix} 2 & 7 & 9 & -1 \\ 0 & -2.5 & -4.5 & 1.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of D is 2.

6. In the end, converting the matrix to echelon form is trying to find if any combination of rows will lead to a sum of 0, which is the definition of linear independence.

Exercise 5.2

1. (c)

$$\begin{aligned} &8\begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix} - 1\begin{vmatrix} 4 & 1 \\ 6 & 3 \end{vmatrix} + 3\begin{vmatrix} 4 & 0 \\ 6 & 0 \end{vmatrix} \\ =&8(0-0) - 1(12-6) + 3(0-0) \\ =&0 - 6 + 0 \\ =&-6 \end{aligned}$$

(e)

$$\begin{aligned} &a\begin{vmatrix} c & a \\ a & b \end{vmatrix} - b\begin{vmatrix} b & a \\ c & b \end{vmatrix} + c\begin{vmatrix} b & c \\ c & a \end{vmatrix} \\ =&a\left(cb - a^2\right) - b\left(b^2 - ac\right) + c\left(ab - c^2\right) \\ =&abc - a^3 - b^3 + abc + abc - c^3 \\ =&-a^3 - b^3 - c^3 + 3abc \end{aligned}$$

(f)

$$\begin{aligned} &x\begin{vmatrix} y & 2 \\ -1 & 8 \end{vmatrix} - 5\begin{vmatrix} 3 & 2 \\ 9 & 8 \end{vmatrix} + 0\begin{vmatrix} 3 & y \\ 9 & -1 \end{vmatrix} \\ =&x(8y+2) - 5(24-18) + 0(-3-9y) \\ =&8xy + 2x - 30 + 0 \\ =&8xy + 2x - 30 \end{aligned}$$

2. $|C_{13}| : 1 + 3 = 4$ is even so $+$
   $|C_{23}| : 2 + 3 = 5$ is odd so $-$

$|C_{33}|$: $3 + 3 = 6$ is even so +
$|C_{41}|$: $4 + 1 = 5$ is odd so –
$|C_{34}|$: $3 + 4 = 7$ is odd so –

3. Minor of a:
$$|M_{11}| = \begin{vmatrix} e & f \\ h & i \end{vmatrix} = ei - fh$$
Cofactor of a:
$$|C_{11}| = (-1)^{1+1}\,|M_{11}| = |M_{11}|$$
Minor of b:
$$|M_{12}| = \begin{vmatrix} d & f \\ g & i \end{vmatrix} = di - fg$$
Cofactor of b:
$$|C_{12}| = (-1)^{3}\,|M_{12}| = -\,|M_{12}|$$
Minor of f:
$$|M_{23}| = \begin{vmatrix} a & b \\ g & h \end{vmatrix} = ah - bg$$
Cofactor of f:
$$|C_{23}| = (-1)^{5}\,|M_{23}| = -\,|M_{23}|$$

6. Minors of third row:
$$|M_{31}| = \begin{vmatrix} 11 & 4 \\ 2 & 7 \end{vmatrix} = 69, \quad |M_{32}| = \begin{vmatrix} 9 & 4 \\ 3 & 7 \end{vmatrix} = 51, \quad |M_{33}| = \begin{vmatrix} 9 & 11 \\ 3 & 2 \end{vmatrix} = -15$$
Cofactors: $|C_{31}| = |M_{31}|, |C_{32}| = -|M_{32}|, |C_{33}| = |M_{33}|$.