ECON 441

Introduction to Mathematical Economics

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Lecture 3: Linear Algebra

Inverse of a Matrix

For a **square** matrix A, it's inverse A^{-1} is defined as:

$$AA^{-1} = A^{-1}A = I$$

Squareness is a *necessary* condition not a *sufficient* condition

If a matrix's inverse exists, it's called a **nonsingular** matrix

Inverse of a Matrix

If an inverse exists, it is unique.

Proof by contradiction. Let's say $B = A^{-1}$ and $C = A^{-1}$. Then,

$$AB = BA = I$$

$$AC = CA = I$$

Pre-multiply both sides by B,

$$BAC = BCA = BI \implies C = B$$

Solution of Linear-Equation System

$$Ax = b$$

Pre-multiply both sides by A^{-1},

$$A^{-1}Ax = A^{-1}b \quad \implies x = A^{-1}b$$

If A is singular, a unique solution does not exist.

Conditions for Nonsingularity

Squareness is *necessary* but not *sufficient*

Sufficient condition for nonsingularity:

> *Rows (or equivalently) columns are linearly independent*

Example.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Conditions for Nonsingularity

Squareness is *necessary* but not *sufficient*

Sufficient condition for nonsingularity:

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Example.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

A is singular, B is nonsingular.

Conditions for Nonsingularity

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad d = \begin{bmatrix} a \\ b \end{bmatrix}$$

We have a system of linear equations:

$$Ax = d$$

Then,

$$x_1 + 2x_2 = a$$
$$2x_1 + 4x_2 = b$$

Conditions for Nonsingularity

$$x_1 + 2x_2 = a$$
$$2x_1 + 4x_2 = b$$

For these equations to be consistent, we need $b = 2a$:

$$x_1 + 2x_2 = a$$

$$2x_1 + 4x_2 = 2a$$

Both are the same equation, infinite number of solutions.

Conditions for Nonsingularity

To summarize, for a matrix to be nonsingular (i.e. its inverse exists):

Necessary condition: **Squareness**

Sufficient condition: **Rows or (equivalently) columns are linearly independent**

Rank of a Matrix

Rank of a matrix $=$ maximum number of linearly independent rows

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

Rank of A? Rank of B?

Rank of a Matrix

Rank of a matrix $=$ maximum number of linearly independent rows

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

Rank of A? Rank of B?

Full rank $\iff$ Nonsingularity

Checking for Linear Independence

Echelon form of a matrix.

- First row: all elements *can be* non-zero
- Second row: first element 0
- Third row: first two elements 0

$$\vdots$$

- Last row: first $m-1$ elements zero

Checking for Linear Independence

Echelon form of a 2×2 matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$$

Checking for Linear Independence

Echelon form of a 3×3 matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

Checking for Linear Independence

Valid operations to convert to echelon form:

- Interchange any two rows
- Multiplication (or division) of a row by a scalar $k \neq 0$
- Addition of a (or k times of a) row to another

Converting to Echelon Form

Given matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Target elements in order:

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{bmatrix} \rightarrow \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & c_{32} & c_{33} \end{bmatrix} \rightarrow \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ 0 & d_{22} & d_{23} \\ 0 & 0 & d_{33} \end{bmatrix}$$

Checking for Linear Independence

Convert to *echelon* form to check for linear independence.

Example.

$$A = \begin{bmatrix} 0 & -1 & -4 \\ 3 & 1 & 2 \\ 6 & 1 & 0 \end{bmatrix}$$

Checking for Linear Independence

Echelon form, similar to solving by substitution.

In our original example,

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \quad x = \begin{bmatrix} q \\ p \end{bmatrix} \quad b = \begin{bmatrix} 100 \\ 20 \end{bmatrix}$$

Checking for Linear Independence

Consider augmented matrix:

$$A = \left[\begin{array}{cc|c} 1 & 2 & 100 \\ 1 & -3 & 20 \end{array}\right]$$

Reduce to echelon form:

$$A = \left[\begin{array}{cc|c} 1 & 2 & 100 \\ 0 & -5 & -80 \end{array}\right]$$

$$q + 2p = 100 \qquad -5p = -80$$

Checking for Nonsingularity

Rank of a matrix $=$ maximum number of linearly independent rows or (equivalently) columns

If a square matrix has full rank, it is nonsingular.

To check for nonsingularity or finding rank: echelon form.

Alternatively, calculate the **determinant** to check for nonsingularity. For singular matrices, the determinant is zero.

Determinant

Determinant $|A|$ is a unique scalar associated with a *square* matrix A.

Determinant of a 2×2 Matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Can be calculated as:

$$|A| = a_{11}a_{22} - a_{12}a_{21}$$

Determinant of a 3×3 Matrix

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determinant of a $n \times n$ Matrix

A *minor* of the element a_{ij}, denoted by $|M_{ij}|$ is obtained by deleting the ith row and jth column.

Cofactor C_{ij} is defined as:

$$|C_{ij}| = (-1)^{i+j}|M_{ij}|$$

Then,

$$|A| = \sum_{i=1}^{n} a_{ij}|C_{ij}| = \sum_{j=1}^{n} a_{ij}|C_{ij}|$$

References and Homework Problems

- New references for today: 5.1, 5.2
- Homework problems:
  - Exercise 5.1: 3, 4, 5, 6
  - Exercise 5.2: 1 (c) (e) (f), 2, 3, 6