ECON 441

Introduction to Mathematical Economics

Div Bhagia

Lecture 1: Preliminaries

Today's Topics & References

- Numbers and sets (sections 2.2 and 2.3)
- Relations and functions (sections 2.4-2.6, page 163)
- Summation notation (handout)
- Necessary and sufficient conditions (beginning of 5.1)

Numbers and Sets

Real-Number System

- *Integers*:

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

Real-Number System

- *Integers*:

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

- *Fractions*:

$$\frac{1}{2}, \frac{3}{5}, -\frac{2}{3}$$

Real-Number System

- *Integers*:

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

- *Fractions*:

$$\frac{1}{2}, \frac{3}{5}, -\frac{2}{3}$$

- *Rational numbers*: ratio of integers

Real-Number System

- *Integers*:

$$\dots, -3, -2, -1, 0, 1, 2, 3, \dots$$

- *Fractions*:

$$\frac{1}{2}, \frac{3}{5}, -\frac{2}{3}$$

- *Rational numbers*: ratio of integers

  Are fractions rational numbers? What about integers?

Real-Number System

- *Rational numbers*: ratio of integers
  "terminating or repeating decimal"

$$\text{Example. } \frac{1}{3} = 0.333, \frac{1}{4} = 0.25$$

Real-Number System

- *Rational numbers*: ratio of integers
  "terminating or repeating decimal"

  $$\text{Example. } \frac{1}{3} = 0.333, \frac{1}{4} = 0.25$$

- *Irrational numbers*: cannot be expressed as a ratio of two integers
  "nonterminating or nonrepeating decimal"

  $$\text{Example. } \sqrt{2} = 1.4142, \pi = 3.1415$$

Real-Number System

- *Rational numbers*: ratio of integers
  "terminating or repeating decimal"

  Example. $\frac{1}{3} = 0.333, \frac{1}{4} = 0.25$

- *Irrational numbers*: cannot be expressed as a ratio of two integers
  "nonterminating or nonrepeating decimal"

  Example. $\sqrt{2} = 1.4142, \pi = 3.1415$

- *Real numbers ($\mathbb{R}$): rational and irrational*

Sets

- A *set* is a collection of distinct objects.

$$A = \{brownies, icecream, pizza, ramen\}$$

- $pizza \in A$, $\in$ stands for 'is in'

Sets

- A *set* is a collection of distinct objects.

$$A = \{brownies, icecream, pizza, ramen\}$$

- $pizza \in A$, $\in$ stands for 'is in'

- What about sets B and C?

$$B = \{x | x \text{ is a positive integer}\}$$

$$C = \{x | 1 < x < 5\}$$

Set Relations

1. Equivalence ($=$)

$$A = \{brownies, icecream, pizza, ramen\}$$

$$B = \{pizza, ramen, icecream, brownies\}$$

$$A = B$$

Set Relations

1. Equivalence ($=$)

$$A = \{brownies, icecream, pizza, ramen\}$$

$$B = \{pizza, ramen, icecream, brownies\}$$

$$A = B$$

2. Subset ($\subset$)

$$C = \{pizza, ramen\}$$

   $C \neq A$ but $C \subset A$.
   Note: $A \supset C$ is equivalent.

Set Relations

1. Equivalence ($=$)

$$A = \{brownies, icecream, pizza, ramen\}$$

$$B = \{pizza, ramen, icecream, brownies\}$$

$$A = B$$

2. Subset ($\subset$)

$$C = \{pizza, ramen\}$$

$C \neq A$ but $C \subset A$.
Note: $A \supset C$ is equivalent.

Is $A \subset B$?

Set Relations

1. Equivalence ($=$)

$$A = \{brownies, icecream, pizza, ramen\}$$

$$B = \{pizza, ramen, icecream, brownies\}$$

$$A = B$$

2. Subset ($\subset$)

$$C = \{pizza, ramen\}$$

   $C \neq A$ but $C \subset A$.
   Note: $A \supset C$ is equivalent.

   Is $A \subset B$? Yes, but C is a proper subset of A.

Set Relations

3. Disjoint sets

$$A = \{brownies, icecream, pizza, ramen\}$$

$$D = \{salad, fruits\}$$

Set Relations

3. Disjoint sets

$$A = \{brownies, icecream, pizza, ramen\}$$

$$D = \{salad, fruits\}$$

4. Neither but still related

$$A = \{brownies, icecream, pizza, ramen\}$$

$$E = \{salad, fruits, icecream\}$$

Set Relations

- $\emptyset$: empty or null set

Set Relations

- $\varnothing$: empty or null set

- What are all possible subsets of

$$S = \{a, b, c\}$$

Set Relations

- $\emptyset$: empty or null set

- What are all possible subsets of

$$S = \{a, b, c\}$$

$$\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}$$

Set Relations

- $\emptyset$: empty or null set

- What are all possible subsets of

$$S = \{a, b, c\}$$

$\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}$

Always 2^n subsets. Here $n = 3$, so 8 subsets.

Set Operations

1. *Union:* $A \cup B$, elements in either A *or* B
2. *Intersection:* $A \cap B$, elements in both A *and* B

Example:

$$A = \{brownies, icecream, pizza, ramen\}$$

$$B = \{salad, fruits, icecream\}$$

$A \cup B =$

$A \cap B =$

Set Operations

1. *Union:* $A \cup B$, elements in either A or B
2. *Intersection:* $A \cap B$, elements in both A and B

What about

$$A = \{brownies, icecream, pizza, ramen\}$$

$$B = \{salad, fruits\}$$

$A \cup B =$

$A \cap B =$

Set Operations

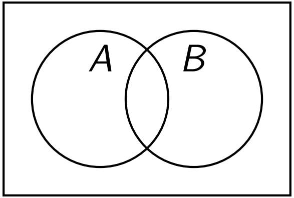
3. *Complement of A*: $\tilde{A}$, 'not A'

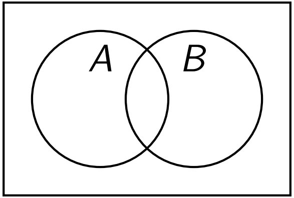
   Universal set U (context specific) then:

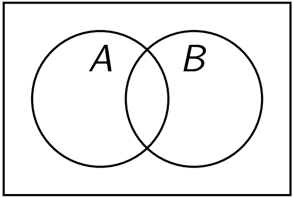
$$\tilde{A} = \{x | x \in U \text{ and } x \notin A\}$$

   Example. $U = \{1, e, f, 2\}$, $A = \{1, 2\}$, then $\tilde{A} = \{e, f\}$.

Set Operations: Venn Diagrams







Laws of Set Operations

- Commutative law

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

- Distributive law

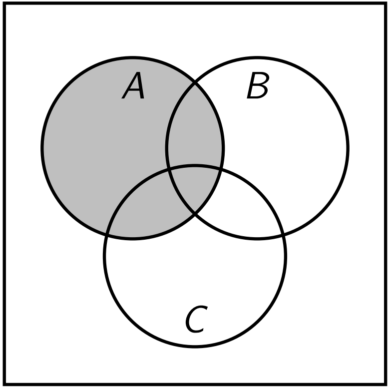
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

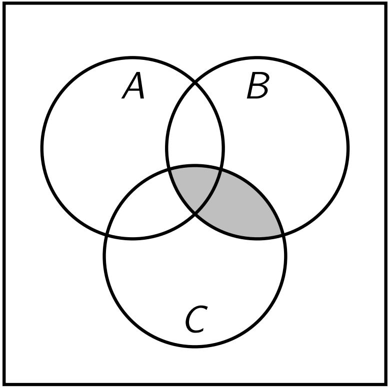
Distributive law

$$\boxed{A \cup (B \cap C)} = (A \cup B) \cap (A \cup C)$$

A



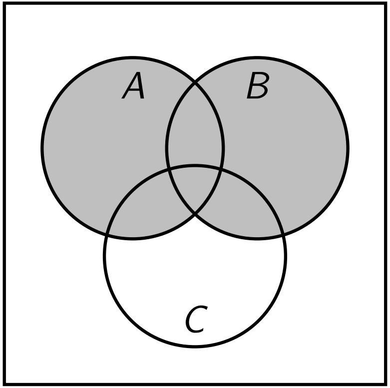
$B \cap C$



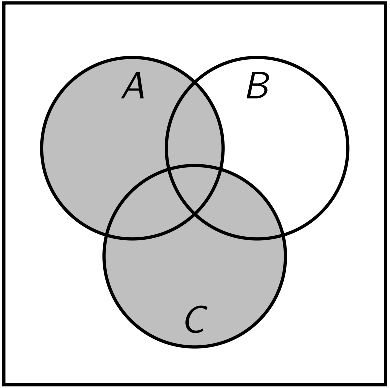
Distributive law

$$A \cup (B \cap C) = \boxed{(A \cup B) \cap (A \cup C)}$$

$A \cup B$



$A \cup C$



Ordered Sets

- We said order does not matter for sets
- But we can have ordered sets where

$$(a, b) \neq (b, a) \text{ unless } a = b$$

- Ordered pairs, triples,...

Example. $(age, weight)$, $(22, 120)$ different from $(120, 22)$

Cartesian Product

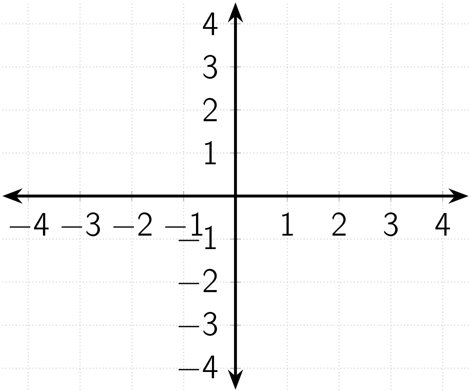
$$A = \{1, 2\} \quad B = \{3, 4\}$$

Cartesian Product: set of all possible ordered pairs

$$A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$$

Cartesian Plane

$$\mathbb{R}^2 = \{(x, y) | x \in \mathbb{R}, y \in \mathbb{R}\}$$



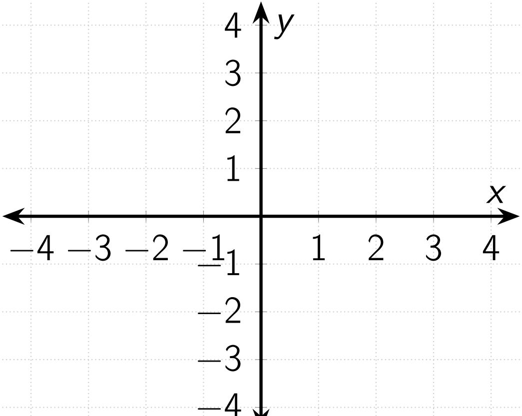
Can have $\mathbb{R}^3, \mathbb{R}^4, \ldots, \mathbb{R}^n$

Relations and Functions

Relations

Relation: subset of the Cartesian product

Example. $\{(x, y) | y \leqslant x\}$



Functions

Function: a relation where for each x there is a unique y

$$f : X \to Y, \quad y = f(x)$$

Examples. $y = x, y = x^2, y = 2x + 3$

X : domain, Y : codomain, $f(X)$: range

Most functions we will encounter, $f : \mathbb{R}^k \to \mathbb{R}$

Functions

Let's say,

$$f : X \to \mathbb{R}, \quad y = 3x - 5$$

where $X = \{2, 3, 4\}$.

What is the range?

Cost Function

Consider the total cost C of producing hats Q,

$$C = f(Q)$$

- Should I be able to produce 10 hats and 5 hats at the same cost?
- Possible to produce 5 hats for \$20 and \$25?

By the way

Consider the total cost C of producing hats Q,

$$C = f(Q) = 2Q + 5$$

What is the cost of producing 1 hat?

What is the cost of producing 2 hats?

How many hats can I produce for $25?

Types of Functions

- Constant: $y = f(x) = 5$
- Polynomial of degree n

  $n = 0$, constant
  $n = 1$, linear
  $n = 2$, quadratic
  $n = 3$, cubic
  ...
- Rational function: ratio of two polynomial functions:

$$y = \frac{a}{x}$$

Function of More than One Variables

Functions can be of two variables:

$$z = g(x, y)$$

Or three, or four,..., or n

Monotonic functions

Strictly increasing function:

$$x_1 > x_2 \rightarrow f(x_1) > f(x_2)$$

Strictly decreasing function:

$$x_1 > x_2 \rightarrow f(x_1) < f(x_2)$$

Increasing function:

$$x_1 > x_2 \rightarrow f(x_1) \geqslant f(x_2)$$

Decreasing function:

$$x_1 > x_2 \rightarrow f(x_1) \leqslant f(x_2)$$

Inverse of a function

Function $y = f(X)$ has an inverse if it is a one-to-one mapping, i.e. each value of y is associated with a unique value of x.

Inverse function

$$x = f^{-1}(y)$$

returns the value corresponding value of x for each y.

One-to-one mapping unique to strictly monotonic functions

Inverse of a function

Example: Find the inverse of $y = f(x) = 3x - 2$.

By the way

What is $x \times x$?

What is $x^2 \times x$?

What is $x^2 \times x^2$?

More generally, $x^n \times x^m = x^{m+n}$

Summation Notation

Summation Notation

$$\sum_{i=1}^{N} x_i = x_1 + x_2 + ... + x_N$$

Example:

$x = \{2, 9, 6, 8, 11, 14\}$

$\sum_{i=1}^{4} x_i = x_1 + x_2 + x_3 + x_4 = 2 + 9 + 6 + 8 = 25$

Summation Notation

Another way of using a summation sign is to write

$$\sum_{x \in A} x$$

which refers to summing up all elements in A.

To sum up x for all possible values x, we can simply write

$$\sum_{x} x$$

Things you CAN do

1. Pull constants out of or into the summation sign.

$$\sum_{i=1}^{N} bx_i = b \sum_{i=1}^{N} x_i$$

Things you CAN do

2. Split apart (or combine) sums (addition) or differences (subtraction)

$$\sum_{i=1}^{N}(bx_i + cy_i) = b\sum_{i=1}^{N}x_i + c\sum_{i=1}^{N}y_i$$

Things you CAN do

3. Multiply through constants by the number of terms in the summation

$$\sum_{i=1}^{N}(a + bx_i) = aN + b\sum_{i=1}^{N} x_i$$

Things you CANNOT do

1. Split apart (or combine) products (multiplication) or quotients (division).

$$\sum_{i=1}^{N} x_i y_i \neq \sum_{i=1}^{N} x_i \times \sum_{i=1}^{N} y_i$$

Things you CANNOT do

2. Move the exponent out of or into the summation.

$$\sum_{i=1}^{N} x_i^a \neq \left(\sum_{i=1}^{N} x_i\right)^a$$

Necessary and Sufficient Conditions

Necessary vs. Sufficient Conditions

q is a necessary condition for p if:

$$p \implies q$$

Necessary vs. Sufficient Conditions

q is a necessary condition for p if:

$$p \implies q$$

p: I ate tofu for dinner

q: My dinner had protein

Necessary vs. Sufficient Conditions

q is a sufficient condition for p if:

$$p \Longleftarrow q$$

Necessary vs. Sufficient Conditions

q is a sufficient condition for p if:

$$p \Longleftarrow q$$

p: A number is even

q: A number is divisible by 4

Necessary vs. Sufficient Conditions

q is both necessary and sufficient for p

$$p \iff q$$

Necessary vs. Sufficient Conditions

q is both necessary and sufficient for p

$$p \iff q$$

p: A number is even

q: A number is divisible by 2

Necessary vs. Sufficient Conditions

p: It is a holiday

q: It is Thanksgiving

Necessary vs. Sufficient Conditions

p: The car is out of gas

q: The car isn't starting

Necessary vs. Sufficient Conditions

p: A geometric figure has four sides

q: It is a rectangle

Homework Questions

- Exercise 2.3: 1, 2
- Exercise 2.4: 5, 7, 8
- Exercise 2.5: 1 (For each part, find the inverse of the function too.)
- Exercise 4.2: 6, 8
- Exercise 5.1: 1