#### ECON 441

#### Introduction to Mathematical Economics

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**Final Exam Review** 

# Numbers, Sets, and Functions

- Types of numbers: integers, fractions, rational numbers, irrational numbers, real numbers.
- Set notation:

Example:  $A = \{a, b, c, d\}$  or  $A = \{x | x \in \mathbb{R}\}$ 

- Set relations: equivalence, subset, disjoint
- Set operations: union, intersection, complement

#### Numbers, Sets, and Functions

• Cartesian product

Example:  $\mathbb{R}^2 = \{(x, y) | x \in \mathbb{R}, y \in \mathbb{R}\}$ 

- Relation: subset of a Cartesian product
- Function: a relation where for each *x* there is a unique *y*

$$f: X \to Y, \quad y = f(x)$$

X : domain, Y : codomain, f(X) : range

# Numbers, Sets, and Functions

- One-to-one function: each value of *y* is also associated with a unique value of *x*
- One-to-one mapping unique to strictly monotonic functions
- Inverse of a function only exists for strictly monotonic functions

$$x = f^{-1}(y)$$

returns the value corresponding value of x for each y.

### **Summation Notation**

Example 1:

 $\sum_{i=1}^{3}\sum_{j\leqslant i}X_{i}Y_{j}$ 

#### **Summation Notation**

Example 2:

 $\sum^{2} \sum^{2} (X_i Y_j + 4Y_j^2 + 1)$ i=1 i=1

- Matrix operations: addition, subtraction, scalar multiplication, matrix multiplication
- Identity matrix, transpose of a matrix
- Inverse of a matrix:  $AA^{-1} = A^{-1}A = I$
- Solution of a linear-equation system Ax = b

$$A^{-1}Ax = A^{-1}b \to x = A^{-1}b$$

• Finding the determinant |A| and inverse of a matrix

$$A^{-1} = \frac{1}{|A|} A dj A$$

- If a matrix's inverse exists, it's called a **nonsingular** matrix
- Necessary and sufficient conditions for nonsingularity:
  - *Necessary*: square matrix
  - *Sufficient*: rows (or equivalently columns) are linearly independent
- Rank of matrix: maximum number of linearly independent rows (square matrix with full rank = nonsingular)
- For singular matrices the determinant |A| = 0

Say we have the following system of equations:

$$3x + 2y = 20$$
$$6x + 4y = 40$$

Can write this as:

$$Av = b$$

where

$$A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} \quad v = \begin{bmatrix} x \\ y \end{bmatrix} \quad b = \begin{bmatrix} 20 \\ 40 \end{bmatrix}$$

Unique solution for this system does not exist as A is singular.

Let's solve the following system of equations

$$3x + 2y = 20$$
$$6x - 3y = 40$$

- Limit definition of differentiability and continuity
- Rules of differentiation to differentiate functions (including log and exponential functions)
- Partial and total derivatives
- Second-order derivatives
- Elasticities and partial elasticities

For the function:

$$y = f(x_1, x_2, \dots, x_n)$$

#### Note that the gradient and Hessian is given by

$$\nabla f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \qquad H = \begin{bmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ f_{21} & f_{22} & \dots & f_{2n} \\ \vdots & \vdots & & \vdots \\ f_{n1} & f_{n2} & \dots & f_{nn} \end{bmatrix}$$

Calculate the total differential of the production function

Q = F(K, L)

to find how a small change in both labor and capital affects the production.

Consider a company that allocates its marketing budget X(a) based on the economic climate, represented by an economic index *a*.

$$X(a) = 10a$$

The company's sales revenue Y depends on both the marketing spend X(a) and the economic index *a*.

$$Y(X(a), a) = X(a) \cdot \log(1 + a)$$

How does this company's revenue vary with respect to the economic index *a*? 13 / 26

# **Single-Variable Optimization**

• Given a function

$$y = f(x)$$

- Critical point f'(x\*) = 0, necessary condition for an optimum
- Sufficient condition:
  - maximum if *f*"(*x*\*) < 0
  - minimum if  $f''(x^*) > 0$

### Single-Variable Optimization

Let's find the extrema for the following function and plot it:

$$f(x) = x^4 - 2x^2$$

# **Single-Variable Optimization**

Say, f(x) is a strictly concave function and

$$f'(x^*)=0$$

Is  $f(x^*)$  the global maximum? Can you explain why?

#### **Multiple-Variable Optimization**

$$y = f(x_1, x_2, \dots, x_n)$$

First-order condition:

$$\nabla f(x_1, x_2, ..., x_n) = 0$$

That is:

$$f_1(x_1, x_2, ..., x_n) = 0$$
  

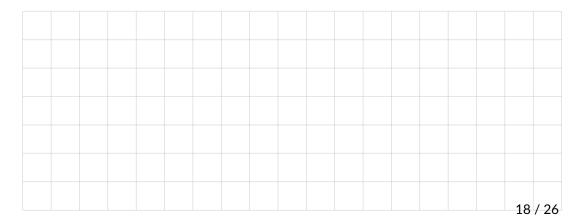
$$f_2(x_1, x_2, ..., x_n) = 0$$
  
:  

$$f_n(x_1, x_2, ..., x_n) = 0$$

#### **Multiple-Variable Optimization**

$$\pi(K, L) = AK^{1/3}L^{2/3} - wL - rK$$

Show that at the optimal: wL = 2rK



### **Envelope Theorem**

Value function:

$$V = f(x^*(\alpha), y^*(\alpha), \alpha)$$

If we differentiate *V* with respect to  $\alpha$ :

$$\frac{dV}{d\alpha} = f_x^* \cdot \frac{dx^*}{d\alpha} + f_y^* \cdot \frac{dy^*}{d\alpha} + f_\alpha^*$$

From the first order conditions we know  $f_x^* = f_y^* = 0$ , therefore

$$\frac{dV}{d\alpha} = f_{\alpha}^*$$

#### **Envelope Theorem**

$$V = \pi(K^*, L^*)$$

How does optimal profit change due to a change in w or r?

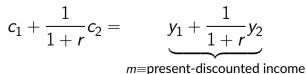
#### **Constrained Optimization**

$$U = U(c_1, c_2) = \ln c_1 + \beta \ln c_2$$
  $0 < \beta < 1$ 

- $y_1, y_2 > 0$ : income in period 1 and 2
- Income you save *s* in period 1 earns interest *r* > 0
- In which case,

$$c_1 + s = y_1$$
  $c_2 = y_2 + (1 + r)s$ 

• Combining these constraints:



# **Constrained Optimization**

$$\max_{\{c_1,c_2\}} \quad U(c_1,c_2) = \ln c_1 + \beta \ln c_2 \quad s.t. \quad c_1 + \frac{1}{1+r}c_2 = m$$



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# **Envelope Theorem with Constraints**

Value function:

$$V = f(x^*(\alpha), y^*(\alpha), \alpha)$$

By envelope theorem:

$$\frac{dV}{d\alpha} = \frac{\partial L^*}{\partial \alpha}$$

How does the optimal utility change due to changes in r,  $y_1$ ,  $y_2$ , or  $\beta$ ?

#### Interpretation of the Lagrange Multiplier

Lagrangian function:

$$L = f(x, y) + \lambda[c - g(x, y)]$$

Substituting the solutions into the objective function, we get

$$V = f(x^*(c), y^*(c))$$

By the envelope theorem,

$$\frac{dV}{dc} = \frac{\partial L^*}{\partial c} = \lambda^*$$

# **Global Optimizers with Constraints**

Consider the problem:

Maximize  $f(x_1, x_2, ..., x_n)$  subject to  $g(x_1, x_2, ..., x_n) = k$ .

The stationary point  $(x_1^*, x_2^*, ..., x_n^*)$  of the lagrangian is a global maximum if:

- 1.  $f(x_1, x_2, ..., x_n)$  is quasiconcave
- 2. The constraint set is convex

# A Few Last Words

Please fill the SOQs :)

#### Thanks for a great semester.

Good luck and don't be a stranger!