Midterm Spring 2024: Solutions

ECON 441: Introduction to Mathematical Economics — Instructor: Div Bhagia

Print Name: ____

This is a closed-book test. You may not use a phone or a computer.

Time allotted: 110 minutes
Total points: 30

Please show sufficient work so that the instructor can follow your work.

I understand and will uphold the ideals of academic honesty as stated in the honor code.

Signature: ______________________________

1. (6 pts) Answer the following questions.

   (a) (1 pt) Consider a mapping $f(x)$. For two distinct values of x, x_0 and x_1, $f(x_0) = f(x_1)$. Is f a valid function? Answer yes or no. **Yes**

   (b) (2 pts) Find the union and intersection for the following sets:

   $$A = \{x : x \text{ is an even number}\} \qquad B = \{2, 4, 8\}$$

   $$\mathbf{A} \cup \mathbf{B} = \mathbf{A} \qquad \mathbf{A} \cap \mathbf{B} = \mathbf{B}$$

   (c) (1 pt) Consider the following two-variable function:

   $$f(x, y) = x + y$$

   where $x \in (0, 1)$ and $y \in (0, 1)$. What is the range of f?

   $$(\mathbf{0}, \mathbf{2})$$

   (d) (1 pt) Given a system of linear equations $Ax = b$, if $|A| = 5$, what can we say about the solution for this system of equations?

   - ☐ Has no solution.
   - ☑ Has a unique solution.
   - ☐ Has infinitely many solutions.
   - ☐ None of the above

   (e) (1 pt) Is the function $y = |x|$ continuous at $x = 0$? Answer yes or no. **Yes**

2. (5 pts) *Consider the following matrix*

$$A = I - X(X'X)^{-1}X'$$

(a) (3 pts) *Is A a square matrix? Show your work or reasoning that led you to this conclusion.*

Say the dimension of X is $m \times n$. Then the dimension of $X'_{n\times m}X_{m\times n}$ is $n \times n$. So the dimension of $(X'X)^{-1}$ is also $n \times n$. This implies that the dimension of $X_{m\times n}\,(X'X)^{-1}_{n\times n}\,X'_{n\times m}$ is $m \times m$. Hence, $X'X$ and A must be square matrices, but X need not be square.

(b) (2 pts) *Prove that A is idempotent i.e. $AA = A$.*

$$AA = I - X\,(X'X)^{-1}\,X' - X\,(X'X)^{-1}\,X' + X\,\underbrace{(X'X)^{-1}\,X'X}_{I}\,(X'X)^{-1}\,X'$$

$$= I - X\,(X'X)^{-1}\,X' = A$$

3. (8 pts) *Consider the following system of equations:*

$$x - 2z = 2$$
$$y + z = 12$$
$$x + y + z = 24$$

(a) (1 pt) *Write this system of equations in matrix format i.e.,*

$$Av = b$$

What is A, v, and b equal to?

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 12 \\ 24 \end{bmatrix}$$

(b) (2 pts) *Calculate the adjoint of A.*

We first need to calculate all the cofactors of A.

$$|C_{11}| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0 \quad |C_{12}| = -1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1 \quad |C_{13}| = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$$

$$|C_{21}| = -1\begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix} = -2 \quad |C_{22}| = \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = 3 \quad |C_{23}| = -1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1$$

$$|C_{31}| = \begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix} = 2 \quad |C_{32}| = -1\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = -1 \quad |C_{33}| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\text{Adj } A = \begin{bmatrix} 0 & -2 & 2 \\ 1 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

(c) (2 pts) *Calculate the determinant of A. Is A nonsingular?*

$$\begin{aligned} |A| &= a_{11}\,|c_{11}| + a_{12}\,|c_{12}| + a_{13}\,|c_{13}| \\ &= 1.0 + 0.1 + (-2)\cdot(-1) = 2 \end{aligned}$$

A is nonsingular as $|A| \neq 0$.

(d) (1 pt) *If you premultiply A^{-1} on both sides of the equation $Av = b$, you should be able to derive an expression to solve for v. Write down this expression.*

Premultiplying by A^{-1} :

$$A^{-1}Av = A^{-1}b$$

Since $A^{-1}A = I$, we have $v^* = A^{-1}b$.

(e) (2 pts) *Using the expression in (d) solve for v^*.*

Since, $A^{-1} = \frac{1}{|A|} Adj A$

$$v^* = \frac{1}{2} \begin{bmatrix} 0 & -2 & 2 \\ 1 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 12 \\ 24 \end{bmatrix}_{3\times 1}$$

$$= \frac{1}{2} \begin{bmatrix} -24+48 \\ 2+36-24 \\ -2-12+24 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \\ 5 \end{bmatrix}$$

Checking if it's correct:

$$12 - 2(5) = 2, \quad 7 + 5 = 12, \quad 12 + 7 + 5 = 24$$

4. (4 pts) *Differentiate the following functions:*

(a)

$$y = 3x^3 + x^2 + 4, \quad \frac{dy}{dx} = 9x^2 + 2x$$

(b)

$$y = \frac{1}{x} + 3x^2, \quad \frac{dy}{dx} = \frac{-1}{x^2} + 6x$$

(c)

$$y = \frac{x-1}{x^2+3}, \quad \frac{dy}{dx} = \frac{1\left(x^2+3\right) - 2x(x-1)}{\left(x^2+3\right)^2} = \frac{-x^2+2x+3}{\left(x^2+3\right)^2}$$

5. (5 pts) *Here is a demand function:*

$$Q = 100 - 0.4p$$

where Q is the quantity demanded and p is the price.

(a) *Calculate the elasticity of demand ε in terms of p.*

$$\varepsilon = \frac{dQ}{dp} \cdot \frac{p}{Q} = \frac{-0.4p}{100 - 0.4p}$$

(b) *What is the elasticity at $p = 50$? What about at $p = 100$? Is demand elastic ($|\varepsilon| > 1$) or inelastic ($|\varepsilon| < 1$) at these prices?*

At $p = 50, \varepsilon = -\frac{1}{4} = -0.25$
At $p = 100, \varepsilon = -\frac{2}{3} = -0.66$
Demand is inelastic at these prices.

(c) *Is the elasticity monotonically decreasing or increasing with price? (Note: I suggest taking the derivative of ε with respect to p instead of guessing.)*

$$\begin{aligned}\frac{d\varepsilon}{dp} &= \frac{-0.4(100 - 0.4p) + 0.4(-0.4p)}{(100 - 0.4p)^2} \\ &= \frac{-40 + 0.16p - 0.16}{(100 - 0.4p)^2} \\ &= \frac{-40}{(100 - 0.4p)^2} < 0\end{aligned}$$

ε is monotonically decreasing in price. Higher the price, more elastic the demand is.

6. (2 pts) *Say we have the following relationship between income (Y), consumption (C), and saving (S).*

$$Y = C + S$$

In addition, saving depends on interest rate i as follows:

$$S = g(i) + 100$$

Find the total derivative of income with respect to the interest rate.

$$\frac{dY}{di} = \frac{dY}{dC} \cdot \underbrace{\frac{dC}{di}}_{0} + \underbrace{\frac{dY}{dS}}_{1} \cdot \underbrace{\frac{dS}{di}}_{g'(i)} = g'(i)$$