Spring 2023 Midterm Exam: Solutions

ECON 441: Introduction to Mathematical Economics — Instructor: Div Bhagia

Print Name: ____

This is a closed-book test. You may not use a phone or a computer.

Time allotted: 110 minutes
Total points: 30

Please show sufficient work so that the instructor can follow your work.

I understand and will uphold the ideals of academic honesty as stated in the honor code.

Signature: ______________________________

1. (8 pts) Answer the following questions (1 point each)

   (a) Consider two sets A and B, where A is the set of all odd real numbers and B is the set of all real numbers. What is the intersection of A and B?

$$A \cap B = A$$

   (b) Expand the following summation expression: $\sum_{i=0}^{3}(x+i)^2$

$$x^2 + (x+1)^2 + (x+2)^2 + (x+3)^2$$

   (c) Find the inverse of $f(x) = \frac{x-2}{3}$.

$$f^{-1}(y) = 3y + 2 \quad \text{or} \quad g(x) = 3x + 2$$

   (d) Why do we need a matrix to be nonsingular when solving systems of linear equations?

   - ☑ To ensure that the system of equations has a unique solution.
   - ☐ To ensure that the system of equations has no solutions.
   - ☐ To ensure that the system of equations has infinitely many solutions.
   - ☐ It does not matter if the matrix is singular or nonsingular.

   (e) Is the following function continuous? Is it differentiable?

$$f(x) = \begin{cases} 4 & \text{if } x < 2 \\ 10 & \text{if } x \geq 2 \end{cases}$$

   Neither continuous, nor differentiable.

(f) For the function $f(x) = \ln x$, $f'(x) = 1/x$

- ☑ True
- ☐ False

(g) Find the derivative of $y = \frac{1}{x}$.

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

(h) Find the derivative of $y = (2 - 3x)(1 + x)$.

$$\frac{dy}{dx} = -3(1 + x) + 1(2 - 3x) = -3 - 3x + 2 - 3x = -(1 + 6x)$$

2. (10 pts) Consider the following system of equations:

$$4x + 3y - 2z = 7$$
$$x + y = 5$$
$$3x + z = 4$$

(a) (1.5 pt) Write this system of equations in matrix format, i.e.,

$$Av = b$$

What is A, v, and b equal to?

$$A = \begin{bmatrix} 4 & 3 & -2 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad b = \begin{bmatrix} 7 \\ 5 \\ 4 \end{bmatrix}$$

(b) (3 pts) Calculate the adjoint of A.

To find the adjoint of a matrix, we need to find the transpose of the matrix of cofactors. Let's first find the 9 cofactors.

$$C_{11} = (-1)^2 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad C_{12} = (-1)^3 \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = -1 \qquad C_{13} = (-1)^4 \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} = -3$$

$$C_{21} = (-1)^3 \begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} = -3 \quad C_{22} = (-1)^4 \begin{vmatrix} 4 & -2 \\ 3 & 1 \end{vmatrix} = 10 \quad C_{23} = (-1)^5 \begin{vmatrix} 4 & 3 \\ 3 & 0 \end{vmatrix} = 9$$

$$C_{31} = (-1)^4 \begin{vmatrix} 3 & -2 \\ 1 & 0 \end{vmatrix} = 2 \qquad C_{32} = (-1)^5 \begin{vmatrix} 4 & -2 \\ 1 & 0 \end{vmatrix} = -2 \quad C_{33} = (-1)^6 \begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix} = 1$$

Then the adjoint of A is given by:

$$AdjA = C' = \begin{bmatrix} 1 & -3 & 2 \\ -1 & 10 & -2 \\ -3 & 9 & 1 \end{bmatrix}$$

(c) (2 pts) Calculate the determinant of A. Is A nonsingular?

To calculate the determinant of A by expanding with respect to the third row:

$$|A| = a_{31}|C_{31}| + a_{32}|C_{32}| + a_{33}|C_{33}| = 3.2 + 0.(-2) + 1.1 = 7$$

Since $|A| \neq 0$, A is nonsingular.

(d) (1.5 pt) If you premultiply A^{-1} on both sides of the equation $Av = b$, you should be able to derive an expression to solve for v. Write down this expression.

$$\underbrace{A^{-1}A}_{I} v = A^{-1}b \rightarrow v^* = A^{-1}b$$

(e) (2 pts) Using the expression in (d) solve for v^*.

Note that,

$$A^{-1} = \frac{1}{|A|}AdjA$$

Then,

$$v^* = A^{-1}b = \frac{1}{7}\begin{bmatrix} 1 & -3 & 2 \\ -1 & 10 & -2 \\ -3 & 9 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 5 \\ 4 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 - 15 + 8 \\ -7 + 50 - 8 \\ -21 + 45 + 4 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 0 \\ 35 \\ 28 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix}$$

3. (6 pts) Fun with Calculus!

(a) (3 pts) Demand for a good as a function of its price is given as follows:

$$Q(p) = p^{-\frac{1}{1+\alpha}}$$

Calculate the elasticity of demand with respect to price. (Note: You can also take the log of both sides of the equation and write $\ln Q = -\frac{1}{1+\alpha} \cdot \ln p$, and use that equation if you like.)

Note that,

$$\frac{dQ}{dp} = -\frac{1}{1+\alpha} \cdot p^{-\frac{1}{1+\alpha}-1}$$

Plugging this and $Q = p^{-\frac{1}{1+\alpha}}$ in the formula for elasticity:

$$\varepsilon = \frac{dQ}{dp} \cdot \frac{p}{Q} = -\frac{1}{1+\alpha} \cdot p^{-\frac{1}{1+\alpha}-1} \cdot \frac{p}{p^{-\frac{1}{1+\alpha}}} = -\frac{1}{1+\alpha}$$

I think my note in the parenthesis confused some of you. I was suggesting that alternatively you could write the equation in logs and find the elasticity as follows:

$$\ln Q = -\frac{1}{1+\alpha} \cdot \ln p$$

Differentiating both sides with respect to p:

$$\frac{dQ}{dp} \cdot \frac{1}{Q} = -\frac{1}{1+\alpha} \cdot \frac{1}{p}$$

Rearrange above equation to bring the p on the left-hand side:

$$\frac{dQ}{dp} \cdot \frac{p}{Q} = -\frac{1}{1+\alpha} = \varepsilon$$

(b) (3 pts) Suppose that aggregate income Y and population P are given by:

$$Y(t) = \ln P(t), \qquad P(t) = ae^{rt}$$

where c, a, and r are constants. t denotes time. Find the growth rate of income, which is given by the derivative of Y with respect to t.

We can find this using the chain rule:

$$\frac{dY}{dt} = \frac{dY}{dP} \cdot \frac{dP}{dt} = \frac{1}{P(t)} \cdot are^{rt} = \frac{1}{ae^{rt}} \cdot are^{rt} = r$$

In the last step we are just plugging in $P(t) = ae^{rt}$.

4. (6 pts) Consider the following production function with two inputs, capital (K) and labor (L):

$$Q = 2K^{1/2}L^{1/2}$$

The marginal product of an input is given by the partial derivative of the production function with respect to that input variable.

(a) (3 pts) Show that the marginal product of capital (MPK) and labor (MPL)for the above production function are given by:

$$MPK = \frac{1}{2} \cdot \frac{Q}{K} \qquad MPL = \frac{1}{2} \cdot \frac{Q}{L}$$

MPK is the partial derivative of Q with respect to K:

$$MPK = \frac{\partial Q}{\partial K} = K^{-1/2}L^{1/2}$$

MPL is the partial derivative of Q with respect to L:

$$MPL = \frac{\partial Q}{\partial L} = K^{1/2}L^{-1/2}$$

To see that the expressions given in the question are the same as the partial derivatives above, we can plug-in $Q = 2K^{1/2}L^{1/2}$ in both expressions as follows:

$$MPK = \frac{1}{2} \cdot \frac{Q}{K} = \frac{1}{2} \cdot \frac{2K^{1/2}L^{1/2}}{K} = K^{-1/2}L^{1/2}$$

$$MPL = \frac{1}{2} \cdot \frac{Q}{L} = \frac{1}{2} \cdot \frac{2K^{1/2}L^{1/2}}{L} = K^{1/2}L^{-1/2}$$

(b) (2 pts) Now, say that in equilibrium, wages (w) are equal to the marginal product of labor i.e.

$$w = \frac{1}{2} \cdot \frac{Q}{L} = K^{1/2}L^{-1/2}$$

Given $K = 100$, write labor demand L as a function of wages w. (Essentially, you are finding the inverse of a function).

With $K = 100$, we have:

$$w = (100)^{1/2}L^{-1/2} \rightarrow L = \frac{100}{w^2}$$

(c) (1 pt) Given your answer in (b), do you think labor demand increases or decreases with an increase in wages?
Labor demand decreases with an increase in wages.