Fall 2023 Midterm Exam

ECON 441: Introduction to Mathematical Economics — Instructor: Div Bhagia

Print Name: ____

This is a closed-book test. You may not use a phone or a computer.

Time allotted: 110 minutes
Total points: 30

Please show sufficient work so that the instructor can follow your work.

I understand and will uphold the ideals of academic honesty as stated in the honor code.

Signature: ______________________________

1. (10 pts) Answer the following questions

   (a) (1 pt) Consider two sets A and B, where A is the set of all odd real numbers and B is the set of all even real numbers. What is the union of A and B?

   (b) (1 pt) Expand the following summation expression:

   $$\sum_{j=1}^{2}\sum_{i=1}^{2} X_i Y_j =$$

   (c) (1pt) Does the inverse of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ exist? If yes, find the inverse.

   (d) (1.5 pts) Is the following function continuous? Is it differentiable? Justify your answer.

   $$f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

(e) (1.5 pts) Find $A'A$ where

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Remember: A' is the transpose of A.

(f) (1 pt) Find the derivative of $f(x) = x^2 \ln x$.

(g) (1 pt) Find the derivative of $f(x) = \dfrac{5e^x}{5x + e^x}$

(h) (1 pt) Find the following integral: $\int e^x dx$

(i) (1 pt) Find the following integral: $\int_0^3 x^2 dx$

2. (8 pts) Two types of cars, gasoline (g) and electric (e), are available in the market. Denote the price of gasoline cars by p_g and the price of electric cars by p_e. Similarly, denote the quantity of gasoline cars as q_g and electric cars as q_e. The supply and demand equations for these cars are as follows:

Supply Equations:

$$q_e^s = 25 + 0.3p_e$$
$$q_g^s = 50 + 0.3p_g$$

Demand Equations:

$$q_e^d = 100 - 0.5p_e + 0.2p_g$$
$$q_g^d = 150 + 0.2p_e - 0.5p_g$$

Note: At equilibrium, supply is equal to demand: $q_e^s = q_e^d = q_e$ and $q_g^s = q_g^d = q_g$.

(a) (2 pts) Write down the <u>four</u> equations that must hold in equilibrium. Rearrange each equation so that all terms with variables are on one side and all constants are on the other. What are the four unknown variables?

(b) (2 pts) Express this system of equations in matrix format as $Ax = b$. Clearly specify what A, x, and b are.

(c) (2 pts) What is the necessary and sufficient condition for a unique solution for this system of equations to exist?

(d) (2 pts) How would you solve this system of equations using tools from matrix algebra? There's no need to actually solve it; just explain the steps involved.

3. (5 pts) Consider a utility function for two goods x and y given by:

$$U(x, y) = x^{\frac{1}{3}} y^{\frac{2}{3}}$$

The marginal utility of each good is given by the partial derivative of the utility function with respect to that good.

(a) (3 pts) Compute the marginal utility of x (denoted by U_x) and the marginal utility of y (denoted by U_y) for this utility function.

(b) (2 pts) Is the marginal utility of x increasing with respect to x? What about with respect to y?

4. (7 pts) Fun with Calculus!

   (a) (3.5 pts) Let aggregate wealth W, technology A, and population N be defined as:

   $$W(t) = A(t) + N(t), \quad N(t) = ke^{st}, \quad A(t) = at$$

   where k, s, and a are constants, and t represents time. Determine the total rate of change of aggregate wealth with respect to time. (You need to find the total derivative of W with respect to time.)

   (b) (3.5 pts) Denote the demand for a good as a function of its price by $Q(p)$. The following expression holds:

   $$\ln Q(p) = f(p)$$

   Express the elasticity of demand $Q(p)$ with respect to price in terms of $f'(p)$ and p.