# Fall 2022 Midterm Exam: Solutions

ECON 441: Introduction to Mathematical Economics — Instructor: Div Bhagia

Print Name: ____________________________________________

This is a closed-book test. You may not use a phone or a computer.

Time allotted: 110 minutes
Total points: 30

Please show sufficient work so that the instructor can follow your work.

*I understand and will uphold the ideals of academic honesty as stated in the honor code.*

*Signature*: __________________________

1. (5 pts) Answer the following questions (1 point each)

   (a) The cartesian product of two sets $X$ and $Y$ is defined as:

   $$X \times Y = \{(x, y) | x \in X, y \in Y\}$$

   What is the cartesian product of $X = \{a, b\}$ and $Y = \{2, 1\}$?

   $$X \times Y = \{(a, 2), (b, 2), (a, 1), (b, 1)\}$$

   (b) A matrix's inverse exists if its determinant is equal to 0.

   - ☐ True
   - ☑ False

   (c) The function $f(x) = |x|$ is *differentiable* at $x = 0$.

   - ☐ True
   - ☑ False

   (d) For the function $f(x) = e^x$, $f'(x) = f(x)$

   - ☑ True
   - ☐ False

   (e) What is the derivative of $y = 3x^2 + 2$?

   $$\frac{dy}{dx} = 6x$$

2. (5 pts) Given the vector $x$ and matrix $A$ below, show that $x'Ax$ represents a weighted sum of squares. What is the dimension of $x'Ax$?

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$$

$$\begin{aligned} x'Ax &= \begin{bmatrix} x_1 & x_2 \end{bmatrix}_{1\times 2} \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}_{2\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1} \\ &= \begin{bmatrix} a_{11}x_1 & a_{22}x_2 \end{bmatrix}_{1\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1} \\ &= a_{11}x_1^2 + a_{22}x_2^2 = \sum_{i=1}^{2} a_{ii}{x_i}^2 \end{aligned}$$

Dimension of $x'Ax$ is $1 \times 1$.

3. (4 pts) Say I have a system of $m$ equations with $n$ unknowns.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad &\ \ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n &= b_m \end{aligned}$$

(a) (1 pt) What is the necessary condition for the existence of a unique solution for this system in terms of $m$ and $n$?

*Necessary condition for the existence of a unique solution is that the number of equations is equal to the number of unknowns i.e.* $m = n$.

(b) (1 pt) What is the sufficient condition for the existence of a unique solution for this system?

*Sufficient condition for the existence of a unique solution is that all the equations are linearly independent.*

(c) (2 pts) How would you use the tools learned in linear algebra to solve this system of equations?

I *would start by writing out the above system of equations in matrix format, i.e.*

$$Ax = b$$

*where*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

*Now note that premultiplying* $Ax = b$ by $A^{-1}$ *implies that* $x = A^{-1}b$. *So I would find the inverse of* $A$ *and multiply it with the vector* $b$ *to find the solution to this system of equations.*

4. (6 pts) Find the derivative for the following functions (2 pts each):

(a) $y = \ln(x^2 + 1)$

$$\frac{dy}{dx} = \frac{2x}{x^2 + 1}$$

(b) $y = \dfrac{e^x}{1 + e^x}$

$$\frac{dy}{dx} = \frac{e^x\,(1 + e^x) - e^x e^x}{(1 + e^x)^2} = \frac{e^x}{(1 + e^x)^2}$$

(c) $y = v + v^3$ where $v = x + 1$

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx} = \left(1 + 3v^2\right) \cdot 1 = 1 + 3(x + 1)^2$$

5. (5 pts) Given the consumption function

$$C = 200 + 0.6Y$$

where $C$ is consumption, and $Y$ is income.

(a) (3 pts) Find the income elasticity of consumption $\varepsilon_{CY}$, and determine its sign, assuming $Y > 0$.

$$\varepsilon_{CY} = \frac{dC}{dY} \cdot \frac{Y}{C} = \frac{0.6Y}{200 + 0.6Y} > 0$$

(b) (1 pt) Show that this consumption function is inelastic at all positive income levels.

$$0.6Y < 200 + 0.6Y \to \varepsilon_{CY} < 1$$

(c) (1 pt) What is the income elasticity of consumption when income is equal to \$1000?

$$\varepsilon_{CY} = \frac{0.6 \times 1000}{200 + 0.6 \times 1000} = \frac{600}{800} = \frac{3}{4} = 0.75$$

(d) (1 pt) If income increases by 1% from \$1000 to \$1010, by what percent does consumption increase?

By the definition of elasticity, a 1% increase in income leads to a 0.75% increase in consumption.

6. (5 pts) Given the following function:

$$f(x, y, z) = xyz$$

(a) (2 pts) Find the partial derivatives $f_x$, $f_y$, and $f_z$.

$$f_x = yz, f_y = xz, f_z = xy$$

(b) (1 pt) Find the gradient of $f$.

$$\nabla f = \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix}$$

(c) (1 pt) Find the total differential of $f$. You can denote it by $df$.

$$\begin{aligned} df &= f_x \cdot dx + f_y \cdot dy + f_z \cdot dz \\ &= yz \cdot dx + xz \cdot dy + xy \cdot dz \end{aligned}$$

(d) (1 pt) Find the total derivative of $f$ with respect to $x$?

$$\frac{df}{dx} = f_x + f_y \cdot \frac{dy}{dx} + f_z \cdot \frac{dz}{dx}$$

Since, $\frac{dy}{dx} = 0, \frac{dz}{dx} = 0$

$$\frac{df}{dx} = f_x = yz$$